Imagine you need to determine area of a table
You measure x and y the sides of the table
\[ x = 95.0 \pm 0.5 \text{ cm} \]
\[ y = 190.0 \pm 0.5 \text{ cm} \]
Area = 1.81 m²
What is the uncertainty on the area?

Any quantity determined from measurements will also have an uncertainty!
Consider $i$ measurements of area $A$

Uncertainty of $A$ is given by std deviation of all $i$ measurements

$$A = x \times y$$
$$A_i = A + \Delta A_i = (\bar{x} + \Delta x_i)(\bar{y} + \Delta y_i)$$
$$= \bar{x}\bar{y} + \bar{x}\Delta y_i + \bar{y}\Delta x_i + \Delta x_i\Delta y_i$$

$$\bar{A} = \bar{x}\bar{y}$$
$$\Delta A_i = \bar{x}\Delta y_i + \bar{y}\Delta x_i + \Delta x_i\Delta y_i$$

ignore terms like $\Delta x\Delta y$ if $x$ and $y$ are independent

this is the covariance term

is correlation of errors in $x$ and $y$

$= 0$ if $x, y$ are random & indep.

Consider the table area example:

Area = $x.y$

Fractional error on $A$ is quadratic sum of fractional errors on $x$ & $y$

$$\sigma^2_A = \left(\frac{\partial A}{\partial x}\right)^2 \sigma^2_x + \left(\frac{\partial A}{\partial y}\right)^2 \sigma^2_y$$

$\sigma^2_A = y^2\sigma^2_x + x^2\sigma^2_y$

$$\left(\frac{\sigma_A}{A}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$$

This is the last formula you will need in this course - memorise it!

For $F=function(x,y)$ then

$$\sigma^2_F = \left(\frac{\partial F}{\partial x}\right)^2 \sigma^2_x + \left(\frac{\partial F}{\partial y}\right)^2 \sigma^2_y$$

provided that $x$ and $y$ are independent measurements

Pythagorus’ Theorem is an example of a quadratic sum: $a^2=b^2+c^2$
Why a quadratic sum?

Combined error can never be smaller than on measured quantities

\[ x = 95.0 \pm 0.5 \text{ cm means:} \]

*High probability* true value lies between \( x - \Delta x \) and \( x + \Delta x \)

Note: you do not guarantee that this is true

(we will quantify this probabilistic statement later)

If errors are random and \( x, y \) measured independently then equally probable to measure underestimate OR overestimate in \( y \)

Probability to measure overestimate in \( x \) AND \( y \) simultaneously is small

Independent measurements = subject to only random uncertainties

Let's look at the pendulum example...

\[
T = 2\pi \sqrt{\frac{L}{g}} \quad g = 4\pi^2 \frac{L}{T^2}
\]

\[
\sigma_g^2 = \left( \frac{\partial g}{\partial L} \right)^2 \sigma_L^2 + \left( \frac{\partial g}{\partial T} \right)^2 \sigma_T^2
\]

\[
\sigma_g^2 = \left( \frac{4\pi^2}{T^2} \right)^2 \sigma_L^2 + \left( \frac{-8\pi^2 L}{T^3} \right)^2 \sigma_T^2
\]

\[
\left( \frac{\sigma_g}{g} \right)^2 = \frac{\sigma_L^2}{L^2} + 4 \frac{\sigma_T^2}{T^2}
\]

\[
\frac{\sigma_g}{g} = \sqrt{\frac{\sigma_L^2}{L^2} + 4 \frac{\sigma_T^2}{T^2}}
\]
Consider several simple examples - $F$ is a function of variables $x,y,z...$ and $k$ is a constant

$$F = x + k \quad \sigma_F = \sigma_x \quad \text{absolute error remains the same}$$

$$F = xy \quad \left(\frac{\sigma_F}{F}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2 \quad \text{fractional errors add in quadrature}$$

$$F = kx \quad \frac{\sigma_F}{F} = \frac{\sigma_x}{x} \quad \text{fractional error remains unchanged}$$

$$F = x^n \quad \left(\frac{\sigma_F}{F}\right)^2 = n^2 \left(\frac{\sigma_x}{x}\right)^2 \quad \text{fractional error is scaled by } n$$

All these results derived from general formula

Results always given in terms of absolute / fractional error on measured quantity

Total error is never smaller than measurement error!

---

How about sums and differences?

$$F = x + y \quad \sigma_F^2 = \sigma_x^2 + \sigma_y^2 \quad \text{absolute errors add in quadrature}$$

$$F = x - y \quad \sigma_F^2 = \sigma_x^2 + \sigma_y^2 \quad \text{absolute errors add in quadrature}$$

In both cases error is the same, but $F$ can be very different!

Consider $F=x+y$

$x = 10 \pm 1 \quad y = 9 \pm 2$

$F = 19 \pm \sqrt{(1^2+2^2)}$

$F = 19 \pm 2 \quad \text{number of sig. figs not } \pm 2.23 \quad !$

Consider $F=x-y$

$F = 1 \pm \sqrt{(1^2+2^2)}$

$F = 1 \pm \sqrt{5}$

$F = 1 \pm 2 \quad \text{error is larger than the central value!}$
Formula can be trivially extended to many independent variables, \(x, y, z\ldots\)

\[
\sigma_F^2 = \left( \frac{\partial F}{\partial x} \right)^2 \sigma_x^2 + \left( \frac{\partial F}{\partial y} \right)^2 \sigma_y^2 + \left( \frac{\partial F}{\partial z} \right)^2 \sigma_z^2 \ldots
\]

Note: Largest error contribution comes from largest error source

Can sometimes neglect one error if it is much smaller than others

\[
\frac{\sigma_g}{g} = \sqrt{\frac{\sigma_L^2}{L^2} + 4 \frac{\sigma_T^2}{T^2}}
\]

If \(\frac{\sigma_L}{L} = 1\%\) and \(\frac{\sigma_T}{T} = 5\%\)

then we can neglect error on \(L\).

By inspection we see \(\frac{\sigma_g}{g} = 2 \frac{\sigma_T}{T}\)

Full example involving products, sums, powers & multiplicative factors!

Newton’s equations for distance travelled by object

\[
s = vt + \frac{1}{2} at^2
\]

<table>
<thead>
<tr>
<th>velocity</th>
<th>(v = 200 \pm 10) m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>acceleration</td>
<td>(a = 12 \pm 2) m/s²</td>
</tr>
<tr>
<td>time</td>
<td>(t = 6.0 \pm 0.2) s</td>
</tr>
</tbody>
</table>

\(s = 1416\) m

\[
\sigma_s^2 = \left( \frac{ds}{dv} \right)^2 \sigma_v^2 + \left( \frac{ds}{dt} \right)^2 \sigma_t^2 + \left( \frac{ds}{da} \right)^2 \sigma_a^2
\]

\[
\frac{ds}{dv} = t \quad \frac{ds}{dt} = v + at \quad \frac{ds}{da} = \frac{1}{2} t^2
\]

\[
\sigma_t^2 = (t)^2 \sigma_v^2 + (v + at)^2 \sigma_t^2 + \left(\frac{1}{2} t^2\right)^2 \sigma_a^2
\]

\[
\sigma_s^2 = 6^2 \times 10^2 + (200 + 12 \times 6)^2 \times 0.2^2 + \left(\frac{1}{2} \times 6^2\right)^2 \times 2^2
\]

\[
\sigma_s = \sqrt{3600 + 2959 + 1296} = 88.6
\]

\(s = 1416 \pm 89\) m
Beware: some complex formulae can lead to long calculations
May not be worthwhile - use your judgement
In such cases use:

\[ \sigma_F = \frac{1}{2} \left| F(x - \sigma_x) - F(x + \sigma_x) \right| \]

\[ \sigma_F^2 = \left( \frac{F(x - \sigma_x, y) - F(x + \sigma_x, y)}{2} \right)^2 + \left( \frac{F(x, y - \sigma_y) - F(x, y + \sigma_y)}{2} \right)^2 \]

In other words calculate the value of \( F \) for \( x + \sigma_x \) and for \( x - \sigma_x \) and take half the difference