Scientific Measurement

PHY-103

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Lecture 4 - Error Propagation

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Measurement Uncertainty

Imagine you need to determine area of a table

You measure x and y the sides of the table

x = 95.0 ± 0.5 cm

y = 190.0 ± 0.5 cm

Area = 1.81 m^2

What is the uncertainty on the area?

Any quantity determined from measurements will also have an uncertainty!

Consider i measurements of area A
Uncertainty of A is given by std deviation of all i measurements

$$A = x \times y$$
$$A_i = \bar{A} + \Delta A_i = (\bar{x} + \Delta x_i)(\bar{y} + \Delta y_i)$$
$$= \overline{xy} + \bar{x}\Delta y_i + \bar{y}\Delta x_i + \Delta x_i \Delta y_i$$

$$\bar{A} = \overline{xy}$$
$$\Delta A_i = \bar{x}\Delta y_i + \bar{y}\Delta x_i + \Delta x_i \Delta y_i$$

ignore terms like $\Delta x \Delta y$ if x and y are independent
this is the covariance term
is correlation of errors in x and y
= 0 if x, y are random & indep.

$$\sigma_A^2 = \frac{1}{N-1}\sum_i (\Delta A_i)^2$$
$$= \frac{1}{N-1}\sum_i (\bar{x}\Delta y_i + \bar{y}\Delta x_i)^2$$
$$= \frac{1}{N-1}\sum_i \left(\bar{x}^2 \Delta y_i^2 + \bar{y}^2 \Delta x_i^2 + 2\overline{xy}\Delta x_i \Delta y_i\right)$$

$$\sigma_A^2 = \bar{x}^2 \sigma_y^2 + \bar{y}^2 \sigma_x^2$$
$$\left(\frac{\sigma_A}{A}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$$

For F=function(x,y) then

$$\sigma_F^2 = \left(\frac{\partial F}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \sigma_y^2$$

provided that x and y are independent measurements

This is the last formula you will need in this course - memorise it!

Consider the table area example:
Area = x.y

$$\sigma_A^2 = \left(\frac{\partial A}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial A}{\partial y}\right)^2 \sigma_y^2$$
$$= y^2\sigma_x^2 + x^2\sigma_y^2$$

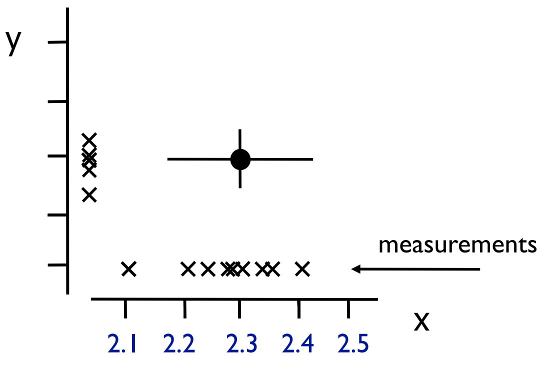
$$\left(\frac{\sigma_A}{A}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$$

Fractional error on A is quadratic sum of fractional errors on x & y

Pythagorus' Theorem is an example of a quadratic sum: $a^2=b^2+c^2$

Why a quadratic sum?

Combined error can never be smaller than on measured quantities



x = 95.0 ± 0.5 cm means:

High probability true value lies between x-Δx and x+Δx

Note: you do not guarantee that this is true

(we will quantify this probabilistic statement later)

If errors are random and x, y measured independently then equally probable to measure underestimate OR overestimate in y

Probability to measure overestimate in x AND y simultaneously is small
Independent measurements = subject to only random uncertainties

Lets look at pendulum example...

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad g = 4\pi^2\frac{L}{T^2}$$

$$\sigma_g^2 = \left(\frac{\partial g}{\partial L}\right)^2 \sigma_L^2 + \left(\frac{\partial g}{\partial T}\right)^2 \sigma_T^2$$

$$\sigma_g^2 = \left(\frac{4\pi^2}{T^2}\right)^2 \sigma_L^2 + \left(\frac{-8\pi^2 L}{T^3}\right)^2 \sigma_T^2$$

$$\left(\frac{\sigma_g}{g}\right)^2 = \frac{\sigma_L^2}{L^2} + 4\frac{\sigma_T^2}{T^2}$$

$$\frac{\sigma_g}{g} = \sqrt{\frac{\sigma_L^2}{L^2} + 4\frac{\sigma_T^2}{T^2}}$$

Consider several simple examples - F is a function of variables $x,y,z...$ and k is a constant

$$F = x + k \qquad \sigma_F = \sigma_x$$ absolute error remains the same

$$F = xy \qquad \left(\frac{\sigma_F}{F}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$$ fractional errors add in quadrature

$$F = kx \qquad \frac{\sigma_F}{F} = \frac{\sigma_x}{x}$$ fractional error remains unchanged

$$F = x^n \qquad \left(\frac{\sigma_F}{F}\right)^2 = n^2 \left(\frac{\sigma_x}{x}\right)^2$$ fractional error is scaled by n

All these results derived from general formula

Results always given in terms of absolute / fractional error on measured quantity

Total error is <u>never</u> smaller than measurement error!

How about sums and differences?

$$F = x + y \qquad \sigma_F^2 = \sigma_x^2 + \sigma_y^2$$ absolute errors add in quadrature

$$F = x - y \qquad \sigma_F^2 = \sigma_x^2 + \sigma_y^2$$ absolute errors add in quadrature

In both cases error is the same, but F can be very different!

Consider $F=x+y$ $\quad x = 10 \pm 1 \quad y = 9 \pm 2$

$F = 19 \pm \sqrt{(1^2+2^2)}$

$F = 19 \pm \sqrt{5}$

$F = 19 \pm 2$ number of sig.figs not ±2.23 !

Consider $F=x-y$

$F = 1 \pm \sqrt{(1^2+2^2)}$

$F = 1 \pm \sqrt{5}$

$F = 1 \pm 2$ error is larger than the central value!

Formula can be trivially extended to many independent variables, x,y,z...

$$\sigma_F^2 = \left(\frac{\partial F}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \sigma_z^2 \ \ldots$$

Note: Largest error contribution comes from largest error source

Can sometimes neglect one error if it is much smaller than others

$$\frac{\sigma_g}{g} = \sqrt{\frac{\sigma_L^2}{L^2} + 4\frac{\sigma_T^2}{T^2}}$$

If $\frac{\sigma_L}{L} = 1\%$ and $\frac{\sigma_T}{T} = 5\%$

then we can neglect error on L

By inspection we see $\frac{\sigma_g}{g} = 2\frac{\sigma_T}{T}$

Full example involving products, sums, powers & multiplicative factors!
Newton's equations for distance travelled by object

$$s = vt + \tfrac{1}{2}at^2$$

velocity v = 200 ± 10 m/s
acceleration a = 12 ± 2 m/s^2
time t = 6.0 ± 0.2 s

s = 1416 m

$$\sigma_s^2 = \left(\frac{ds}{dv}\right)^2 \sigma_v^2 + \left(\frac{ds}{dt}\right)^2 \sigma_t^2 + \left(\frac{ds}{da}\right)^2 \sigma_a^2$$

$$\frac{ds}{dv} = t \qquad \frac{ds}{dt} = v + at \qquad \frac{ds}{da} = \tfrac{1}{2}t^2$$

$$\sigma_s^2 = (t)^2\sigma_v^2 + (v+at)^2\sigma_t^2 + \left(\tfrac{1}{2}t^2\right)^2\sigma_a^2$$

$$\sigma_s^2 = 6^2 \times 10^2 + (200 + 12\times 6)^2 \times 0.2^2 + \left(\tfrac{1}{2}\times 6^2\right)^2 \times 2^2$$

$$\sigma_s = \sqrt{3600 + 2959 + 1296}$$

$$\sigma_s = 88.6$$

s = 1416 ± 89 m

Beware: some complex formulae can lead to long calculations

May not be worthwhile - use your judgement

In such cases use:

$$\sigma_F \approx \tfrac{1}{2}\left|F(x-\sigma_x) - F(x+\sigma_x)\right|$$

$$\sigma_F^2 \approx \left(\frac{\left|F(x-\sigma_x, y) - F(x+\sigma_x, y)\right|}{2}\right)^2 + \left(\frac{\left|F(x, y-\sigma_y) - F(x, y+\sigma_y)\right|}{2}\right)^2$$

In other words calculate the value of F for $x+\sigma_x$ and for $x-\sigma_x$ and take half the difference